J. Fluid Mech. (1993), vol. 254, pp. 313–321

A note on the nonlinear development of the Batchelor–Nitsche instability

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(Received 24 June 1992 and in revised form 6 January 1993)

The nonlinear development of the Batchelor–Nitsche instability (of a periodically stratified fluid) is considered, utilizing the disparity between vertical and horizontal scales of motion. The resulting evolution equation is used to show that the preferred pattern of convection takes the form of rolls, and that the motion evolves to larger and larger horizontal scales as time increases.

1. Introduction

In a recent paper, Batchelor & Nitsche (1991, hereinafter referred to as BN) have identified a previously unremarked instability of a periodically stratified expanse of fluid (for example, one for which the temperature gradient varies sinusoidally in the vertical direction). It turns out that all such structures are unstable in an unbounded fluid, and the instability is dominated by vertical velocities that vary slowly in the horizontal directions. BN only considered the linearized stability problem, and so obtained no information as to the preferred planform of the resulting columnar convection. The calculated discrepancy between vertical and horizontal scales suggests, however, that investigation of the nonlinear development may be made tractable by employing asymptotic methods, such as have been used previously on the problem of convection between poorly conducting boundaries (Chapman & Proctor 1980; Proctor 1981) and for convection (e.g. salt fingering) that takes place in tall thin columns (Proctor & Holyer 1985). In the present paper we adopt an analogous approach, and obtain a coupled pair of partial differential equations in time and the horizontal coordinate. If these equations are linearized, BN's results are recovered in the limit of weak stratification and long horizontal wavelength, while the nonlinear terms can be shown to select rolls (rather than, for example, squares or hexagons) as the preferred mode of convection. Numerical solutions of these equations are also given, with a view to understanding wavenumber selection in the nonlinear regime.

2. Governing equations and scaling

We consider an unbounded region of fluid of kinematic viscosity ν, and thermal diffusivity κ. Gravity $g\hat{z}$ is in the negative z-direction in a Cartesian coordinate system (x, y, z). We assume that conditions are such that the Boussinesq approximation holds. The basic state of the fluid whose instability we propose to investigate is one of zero velocity, and a (time-independent) temperature dependence of the form $T = T_0 - \Delta T \sin z/d$. In common with BN, we must admit that this distribution is somewhat artificial since we are considering diffusive instabilities that evolve no faster than it would take any initial temperature stratification to decay. Indeed, our analysis deals with evolution that occurs slowly on the diffusive timescale. The problem as posed, however, is worth studying as a paradigm of a novel mode of instability; although the precise state envisaged would have to be maintained artificially by some distribution of sources and sinks of heat and/or concentration, it will become unstable in a way that will give a useful guide in more realistic situations. Although our analysis depends on a small quantity ϵ representing the horizontal wavenumber of the disturbance, this is not a parameter whose value is fixed in terms of physical quantities but one that should be thought of as a way of ordering terms in the equations so as to extract useful analytical information, as is regularly done for example in quantum mechanics. We anticipate that our results will remain reasonably accurate even when ϵ is not very small, and so do not regard the existence of the expansion scheme as an important further constraint.

If we denote the fluid velocity by $\kappa \boldsymbol{u}/d$, the total temperature by

$$T = T_0 + \Delta T(\theta - \sin z/d),$$

and non-dimensionalize (x, y, z) with respect to d and time with respect to d^2/κ, then the (dimensionless) equations for $\boldsymbol{u}$ and θ take the form

$$\begin{gathered}\frac{\partial \theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta = \boldsymbol{u}\cdot\hat{\boldsymbol{z}}\cos z + \nabla^2\theta,\\ \frac{1}{\sigma}\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = R\theta\hat{\boldsymbol{z}} - \nabla p + \nabla^2\boldsymbol{u},\\ \nabla\cdot\boldsymbol{u} = 0.\end{gathered} \tag{2.1}$$

Here p is the non-dimensional (reduced) pressure, and the dimensionless constants are the Prandtl number σ and Rayleigh number R defined by

$$\sigma = \frac{\nu}{\kappa}, \quad R = \frac{g\tilde{\alpha}\Delta T d^3}{\kappa\nu}, \tag{2.2}$$

where $\tilde{\alpha}$ is the coefficient of thermal expansion. From now on all quantities will be assumed dimensionless.

BN have considered the linearized version of these equations and are able to show that the latter possesses growing solutions for arbitrarily small R, provided that the horizontal wavelength of the disturbance is sufficiently large. For small horizontal wavenumbers α of order R, the growth rate s of the gravest mode (for which $\boldsymbol{u}\cdot\hat{\boldsymbol{z}}$ and θ are even in z) is given as the positive root of the quadratic

$$(s/\sigma + \alpha^2)(s + \alpha^2) = \tfrac{1}{2}\alpha^2 R^2 \tag{2.3}$$

(cf. BN equation (5.17), in the limit $\alpha \to 0$, $s = O(\alpha^2)$, $R = O(\alpha)$). It is clear from (2.3) that however small R may be, there is a positive growth rate for sufficiently small α.

The result (2.3), together with other results implicit in the details of BN (especially their equation (5.19)), suggests the appropriate scalings for parameters and variables in the long-wavelength limit. We know that R and α are both small in this limit, and of the same order, and so we define the parameter $\epsilon\,(\ll 1)$ as a measure of their size. We might define ϵ to equal R, but it is helpful in the sequel to allow R to vary separately from ϵ; as discussed above we should then think of the latter as a marker labelling orders of smallness.

We make the substitutions

$$\left.\begin{aligned} R = \epsilon r; \quad \partial_t = \epsilon^2 \partial_\tau; \quad (\partial_x, \partial_y) = \epsilon \nabla_H \equiv \epsilon(\partial_\xi, \partial_\eta), \\ \boldsymbol{u} = (\boldsymbol{u}_H, w) = (\epsilon^2 \tilde{\boldsymbol{u}}_H(\boldsymbol{\xi}, z), w_0(\boldsymbol{\xi}) + \epsilon^3 \tilde{w}(\boldsymbol{\xi}, z)); \quad p = \epsilon \tilde{p}, \end{aligned}\right\} \tag{2.4}$$

where $$\boldsymbol{\xi} = (\xi, \eta),$$

and suppose that all quantities depend on the slow time τ. Then the equations become (with $\partial/\partial z \equiv \mathrm{D}$)

$$\epsilon^2 \frac{\partial \theta}{\partial \tau} + w_0 \mathrm{D}\theta + \epsilon^3(\mathrm{D}(\tilde{w}\theta) + \nabla_H \cdot (\tilde{\boldsymbol{u}}_H \theta)) = w_0 \cos z + \epsilon^3 \tilde{w} \cos z + \mathrm{D}^2\theta + \epsilon^2 \nabla_H^2 \theta, \tag{2.5}$$

$$\frac{1}{\sigma}\left[\epsilon^4 \frac{\partial \tilde{\boldsymbol{u}}_H}{\partial \tau} + \epsilon^2 w_0 \mathrm{D}\tilde{\boldsymbol{u}}_H + \epsilon^5(\mathrm{D}(\tilde{w}\tilde{\boldsymbol{u}}_H) + \nabla_H \cdot (\tilde{\boldsymbol{u}}_H \tilde{\boldsymbol{u}}_H))\right]$$
$$= -\epsilon^2 \nabla_H \tilde{p} + \epsilon^2(\mathrm{D}^2 \tilde{\boldsymbol{u}}_H + \epsilon^2 \nabla_H^2 \tilde{\boldsymbol{u}}_H), \tag{2.6}$$

$$\frac{1}{\sigma}\left[\epsilon \frac{\partial w_0}{\partial \tau} + \epsilon^5 \frac{\partial \tilde{w}}{\partial \tau} + \epsilon^3(w_0 \mathrm{D}\tilde{w} + \nabla_H \cdot (\tilde{\boldsymbol{u}}_H w_0) + \mathrm{D}(\tilde{w} w_0)) + O(\epsilon^6)\right]$$
$$= -\epsilon \mathrm{D}\tilde{p} + \epsilon^2 \nabla_H^2 w_0 + \epsilon^3(\mathrm{D}^2 \tilde{w} + \epsilon^2 \nabla_H^2 \tilde{w}) + \epsilon r \theta, \tag{2.7}$$

$$\mathrm{D}\tilde{w} + \nabla_H \cdot (\tilde{\boldsymbol{u}}_H) = 0. \tag{2.8}$$

In the next section we solve these equations sequentially to obtain evolution equations that depend only on $\boldsymbol{\xi}$ and τ. We shall suppose throughout that $\sigma = O(1)$; other, more exotic expansion schemes based (for example) on σ scaling with some power of ϵ are deferred to future work. In principle all the quantities defined in (2.4) should be expanded in powers of ϵ; it turns out, however, that expansion of the velocity variables w_0 and $\tilde{w}$ is not necessary as the final equations involve only their leading-order parts. Note also that the ansatz does not include any z-independent horizontal flows. The scaling adopted here could certainly accommodate such flows, but there is no mechanism for sustaining them at the $O(\epsilon^2)$ level, and if they only appear at higher order than this they have no influence on the leading-order dynamics, provided that $\sigma = O(1)$.

3. The evolution equation

3.1. *Derivation and simple properties*

We begin with equation (2.5); writing $\theta = \theta_0 + \epsilon\theta_1 + \dots$ we obtain at leading order

$$w_0 \mathrm{D}\theta_0 = w_0 \cos z + \mathrm{D}^2\theta_0. \tag{3.1}$$

This is solved by $$\theta_0 = A(\boldsymbol{\xi}, \tau) \cos z + B(\boldsymbol{\xi}, \tau) \sin z, \tag{3.2a}$$

where $$A = w_0/(1 + w_0^2), \quad B = w_0^2/(1 + w_0^2). \tag{3.2b}$$

We notice immediately that the nonlinear term means that the solutions are no longer even about $z = 0$ (though the leading-order velocity field w_0 is independent of z). Then at $O(\epsilon)$ in (2.7) we have (expanding $\tilde{p}$ in powers of ϵ)

$$0 = -\mathrm{D}\tilde{p}_0 + r\theta_0. \tag{3.3}$$

Following BN, we seek disturbances that remain bounded in z and, in particular, require that the z-dependent quantities are actually 2π-periodic in z. As we shall see below, this implies that $\langle \tilde{p}_0 \rangle = 0$, where $\langle \cdot \rangle \equiv (1/2\pi)\int_0^{2\pi} \cdot \, \mathrm{d}z$; then (3.3) is solved by

$$\tilde{p}_0 = r[-B \cos z + A \sin z]. \tag{3.4}$$

The leading-order terms in (2.6) give

$$(1/\sigma)\, w_0 \mathrm{D}\tilde{\boldsymbol{u}}_H = -\boldsymbol{\nabla}_H \tilde{p}_0 + \mathrm{D}^2\tilde{\boldsymbol{u}}_H, \tag{3.5}$$

and so $\tilde{\boldsymbol{u}} = \boldsymbol{\nabla}_H \Phi$, where

$$(1/\sigma)\, w_0 \mathrm{D}\Phi = -\tilde{p}_0 + \mathrm{D}^2\Phi. \tag{3.6}$$

It can now be seen that $\langle \tilde{p}_0 \rangle = 0$ is necessary for Φ to be bounded in z. Thus

$$\Phi = P\cos z + Q \sin z,$$

where

$$P = \frac{r}{1+w_0^2/\sigma^2}\left(B + \frac{w_0 A}{\sigma}\right) = \frac{r w_0^2(1+\sigma)}{\sigma(1+w_0^2)(1+w_0^2/\sigma^2)}, \tag{3.7}$$

$$Q = \frac{r}{1+w_0^2/\sigma^2}\left(-A + \frac{w_0 B}{\sigma}\right) = \frac{r w_0(w_0^2-\sigma)}{\sigma(1+w_0^2)(1+w_0^2/\sigma^2)}.$$

Returning to (2.5) at $O(\epsilon)$, we find

$$w_0 \mathrm{D}\theta_1 = \mathrm{D}^2\theta_1; \tag{3.8}$$

since we are looking for bounded solutions, the only acceptable solution of this equation is $\theta_1 = c(\xi, \tau)$. The equation (2.5) at $O(\epsilon^2)$, $O(\epsilon^3)$ gives

$$\partial\theta_0/\partial\tau - \nabla_H^2\theta_0 = -w_0\mathrm{D}\theta_2 + \mathrm{D}^2\theta_2, \tag{3.9a}$$

$$\partial\theta_1/\partial\tau - \nabla_H^2\theta_1 + \mathrm{D}(\tilde{w}\theta_0) + \boldsymbol{\nabla}_H\cdot(\tilde{\boldsymbol{u}}_H\theta_0) - \tilde{w}\cos z = -w_0\mathrm{D}\theta_3 + \mathrm{D}^2\theta_3. \tag{3.9b}$$

Equation (3.9a) determines the z-dependent part of θ_2 uniquely in terms of θ_0, while (3.9b) possesses the non-trivial solvability condition

$$\frac{\partial c}{\partial \tau} - \nabla_H^2 c + \boldsymbol{\nabla}_H\cdot\langle\tilde{\boldsymbol{u}}_H\theta_0\rangle - \langle\tilde{w}\cos z\rangle = 0. \tag{3.10}$$

Now assuming periodicity in z we have

$$\langle\tilde{w}\cos z\rangle = \langle -\mathrm{D}\tilde{w}\sin z\rangle = \boldsymbol{\nabla}_H\cdot\langle\tilde{\boldsymbol{u}}_H\sin z\rangle = \tfrac{1}{2}\nabla_H^2 Q, \tag{3.11}$$

while

$$\boldsymbol{\nabla}_H\cdot\langle\tilde{\boldsymbol{u}}_H\theta_0\rangle = \tfrac{1}{2}\boldsymbol{\nabla}_H\cdot(A\boldsymbol{\nabla}_H P + B\boldsymbol{\nabla}_H Q). \tag{3.12}$$

Finally, we examine (2.7) at $O(\epsilon^2)$ to obtain

$$\frac{1}{\sigma}\frac{\partial w_0}{\partial \tau} = -\mathrm{D}\tilde{p}_1 + \nabla_H^2 w_0 + r\theta_1; \tag{3.13}$$

since w_0 and θ_1 are independent of z and we require periodic solutions for all quantities including the pressure we take $\tilde{p}_1 = 0$ so that we have

$$\frac{1}{\sigma}\frac{\partial w_0}{\partial \tau} - \nabla_H^2 w_0 = rc. \tag{3.14}$$

Now we write (3.10) in terms of w_0 and c to produce a closed system. Using (3.2) and (3.7) we obtain

$$\frac{\partial c}{\partial \tau} - \nabla_H^2 c + \frac{r}{2\sigma}\boldsymbol{\nabla}_H\cdot\left[\frac{w_0}{1+w_0^2}\boldsymbol{\nabla}_H\left(\frac{w_0^2(1+\sigma)}{(1+w_0^2)(1+w_0^2/\sigma^2)}\right)\right.$$
$$\left. - \frac{1}{1+w_0^2}\boldsymbol{\nabla}_H\left(\frac{w_0(w_0^2-\sigma)}{(1+w_0^2)(1+w_0^2/\sigma^2)}\right)\right] = 0. \tag{3.15}$$

A primitive check on the rather involved sequence of transformations leading to (3.14) and (3.15) is provided by linearizing the latter. We obtain

$$\partial c/\partial\tau - \nabla_H^2 c + \tfrac{1}{2} r \nabla_H^2 w_0 = 0. \tag{3.16}$$

Then, taking (3.14) and (3.16) together, and looking for solutions proportional to $e^{s\tau + i\boldsymbol{k}\cdot\boldsymbol{\xi}}$, we find that

$$(s/\sigma + |\boldsymbol{k}|^2)(s + |\boldsymbol{k}|^2) = \tfrac{1}{2}\gamma^2 |\boldsymbol{k}|^2, \tag{3.17}$$

in agreement with (2.3).

Although (3.15) and (3.16) cannot be called simple they have the great merit, for computational purposes, that they do not depend upon the vertical coordinate z, and in fact the apparently ungainly nonlinear term involves just two derivatives of w_0; thus numerical solution by a finite-difference method presents little difficulty.

3.2. *Weakly nonlinear analysis*

In a layer of infinite horizontal extent there is always a disturbance that is unstable if $r \neq 0$, and the longest wavelength disturbances will always be 'fully nonlinear' in the sense that their spatial structure will not be close to the sinusoidal eigensolution at infinitesimal amplitude. Thus the size of r does not determine the degree of nonlinearity; indeed the size of ϵ was arbitrary in our original formulation, and we could always have chosen ϵ so that $r = 1$. If, however, we fix on a minimum wavenumber for any possible disturbance (simulating the effects of lateral boundary conditions), then for sufficiently small r we can consider weakly nonlinear solutions that are close to sinusoidal. If $|w_0| \ll \min(1, \sigma)$ we may expand the square bracket in (3.15) in powers of w_0; retaining only linear and cubic terms (quadratic terms do not appear) we find that (3.15) can be approximated by

$$\frac{\partial c}{\partial \tau} - \nabla_H^2 c + \frac{r}{2}\nabla_H^2 w_0 - \frac{r}{2\sigma}\left(1 + 2\sigma + \frac{3}{\sigma}\right)\nabla_H \cdot (w_0^2 \nabla_H w_0) = 0. \tag{3.18}$$

As an example of nonlinear selection, consider the evolution of solutions with period $2\pi/\alpha$ in both ξ- and η-directions when $r = r_0 + \delta^2 r_2$ (with δ a further small parameter; we should require in addition that $\delta \gg \epsilon$ so as to avoid consideration of the terms neglected in the derivation of (3.14)–(3.15)), $c = \delta\tilde{c}$, $w_0 = \delta\tilde{w}$, where $r_0 = \alpha\sqrt{2}$ is the critical r for instability at this wavenumber. Then correct to order δ^2

$$\tilde{c} = c_1 e^{i\alpha\xi} + c_2 e^{i\alpha\eta} + \text{c.c.}, \quad \tilde{w} = w_1 e^{i\alpha\xi} + w_2 e^{i\alpha\eta} + \text{c.c.}, \tag{3.19}$$

where the c_i, w_i evolve on the slow timescale $T = \delta^2\tau$. Then we have from (3.17)

$$c_i \approx \frac{\alpha^2}{r_0} w_i - \delta^2\left[\frac{w_{i_T}}{\sigma r_0} + \frac{r_2}{r_0^2}\alpha^2 w_i\right], \quad i = 1, 2, \tag{3.20}$$

and substituting into (3.18) and equating leading-order terms proportional to $e^{i\alpha\xi}$, $e^{i\alpha\eta}$ we obtain the coupled equations

$$\left.\begin{aligned} w_{1_T}\left(1 + \frac{1}{\sigma}\right) - r_2 r_0 w_1 + \frac{F\alpha^2}{\sigma} w_1(|w_1|^2 + 2|w_2|^2) = 0, \\ w_{2_T}\left(1 + \frac{1}{\sigma}\right) - r_2 r_0 w_2 + \frac{F\alpha^2}{\sigma} w_2(|w_2|^2 + 2|w_1|^2) = 0, \end{aligned}\right\} \tag{3.21}$$

where $F = 1 + 2\sigma + 3/\sigma$. It can then be shown that the stable solutions of these equations take the form of rolls, with either w_1 or w_2 vanishing (see e.g. Jenkins &

Proctor 1984). Thus rolls are preferred to squares near onset, at least in periodic boxes, and experience of similar systems suggests that this preference will extend to larger values of r. An exactly analogous calculation can be performed for the interaction of three roll solutions with wavenumbers at a mutual angle of 120° (solutions periodic on a hexagonal lattice). Denoting the velocity amplitudes by w_i, $i = 1,2,3$ we obtain

$$w_{1_T}\left(1+\frac{1}{\sigma}\right)-r_2 r_0 w_1+\frac{F\alpha^2}{\sigma}w_1(|w_1|^2+2|w_2|^2+2|w_3|^2) = 0, \tag{3.22}$$

together with two others obtained by cyclic permutation of the indices. In this case also we can show that the stable solution is of roll, not of hexagon type (that is, all but one of the w_i vanish). The above two cases lend credence to the supposition that roll solutions will be realized in a wide variety of circumstances, and so we shall restrict ourselves in what follows to solutions depending on only one horizontal coordinate.

3.3. *Spatial structure of steady solutions*

If we now restrict our attention to steady solutions depending on the single space variable ξ, we can obtain information about fully nonlinear solutions. From (3.14) we have

$$rc = -\frac{\mathrm{d}^2 w_0}{\mathrm{d}\xi^2}, \tag{3.23}$$

while periodic solutions of (3.15) with zero mean satisfy

$$c = \frac{r}{2\sigma}f(w_0), \tag{3.24}$$

where $$f'(x) = \frac{x}{1+x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2(1+\sigma)}{(1+x^2)(1+x^2/\sigma^2)}\right)-\frac{1}{1+x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x(x^2-\sigma)}{(1+x^2)(1+x^2/\sigma^2)}\right).$$

The closed-form expression for f is complicated in the general case, but for $\sigma = 1$ we find

$$f(x) = \frac{x}{(1+x^2)^2}, \tag{3.25}$$

and the structure of the steady solutions is then given by

$$w_{0\xi\xi}+\frac{r^2}{2}\frac{w_0}{(1+w_0^2)^2} = 0. \tag{3.26}$$

This equation has periodic solutions, which can be expressed if desired in terms of Jacobian elliptic functions. It may be shown that for any L there is a solution of period L for all values of $r^2 > 8\pi^2 L^{-2}$, and these solutions have the property that they are even about each extremum and odd about each zero. Note that for $(N+1)^2 > rL^2/8\pi^2 > N^2$ (N an integer) there are multiple steady solutions with periods Ln^{-1} for $n = 1,2,\ldots,N$. Because of the scale invariance of the system the family of solutions that exists as L varies for fixed r can be mapped on to the family of solutions for fixed L as r varies.

4. Numerical simulations

The system (3.14), (3.15) has been solved numerically for $\sigma = 1$, and w_0 and c functions of (ξ, τ) only, for various values of r. A periodic box with length 2π was taken as the computational domain. In this box the critical value of $|r|$ is $\sqrt{2}$, and for $|r|$

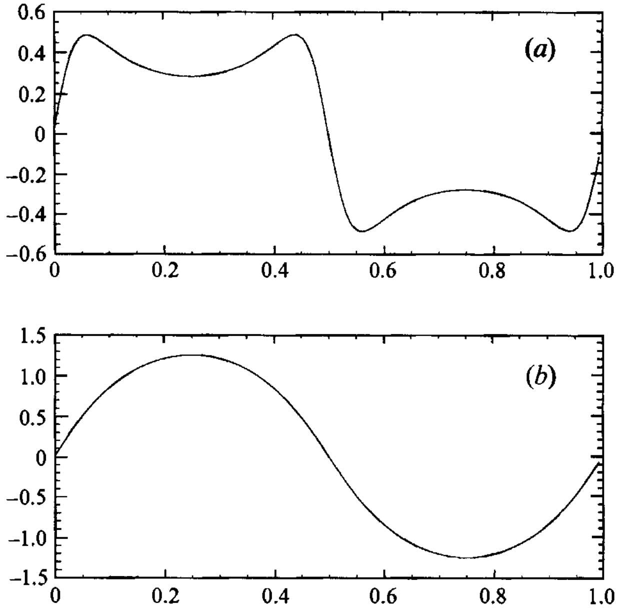

FIGURE 1. Final steady state of roll solutions for $r = 3$ with box length 2π: (*a*), c, (*b*) w.

slightly larger than this value one would expect a final steady state with just one cell pair in the box. For large $|r|$, however, the mode of maximum growth rate will (from (3.17)) have a value of k equal to the integer closest to $|r|/2\surd 2$. In similar systems (e.g. Chapman & Proctor 1980) a value of k close to that for maximum growth rate does indeed dominate the initial evolution, but successive instabilities lengthen the horizontal scale until there is only one cell pair left in the box. In fact, a precisely analogous transition takes place here.

The equations were solved by discretizing on a uniform mesh, and time stepping using a simple explicit Dufort–Frankel scheme. In figure 1 we show the steady state for $r = 3$, for which the long-wavelength mode is the most rapidly growing on linearized theory. The subsidiary extrema of $c(\xi)$ are explained by the identification (3.24) in the steady case, together with the fact that $f(x)$ (equation (3.25)) has a maximum as a function of x. Figure 2 shows a run with $r = 10$, and the initial condition

$$w_0 = 0.01(\sin x + \sin 2x), \; c = 0.$$

In this case the $\sin 2x$ mode is the most rapidly growing and the solution quickly reaches an intermediate state which *nearly* obeys the symmetry of the pure $\sin 2x$ solution. After a relatively long quasi-static period, however, the secondary extrema disappear and we are left with a mode obeying the $\sin x$ symmetry, which resembles that for $r = 3$ but with more extreme boundary-layer features.

5. Discussion

In this short paper have reduced the study of a subset of the weakly nonlinear, long horizontal wavelength motions that arise from the BN instability to a set of coupled partial differential equations in the horizontal coordinates only. We could also have included slow vertical variation of all quantities, but this would have led to a system scarcely simpler than the full nonlinear equations of motion and heat conduction; in

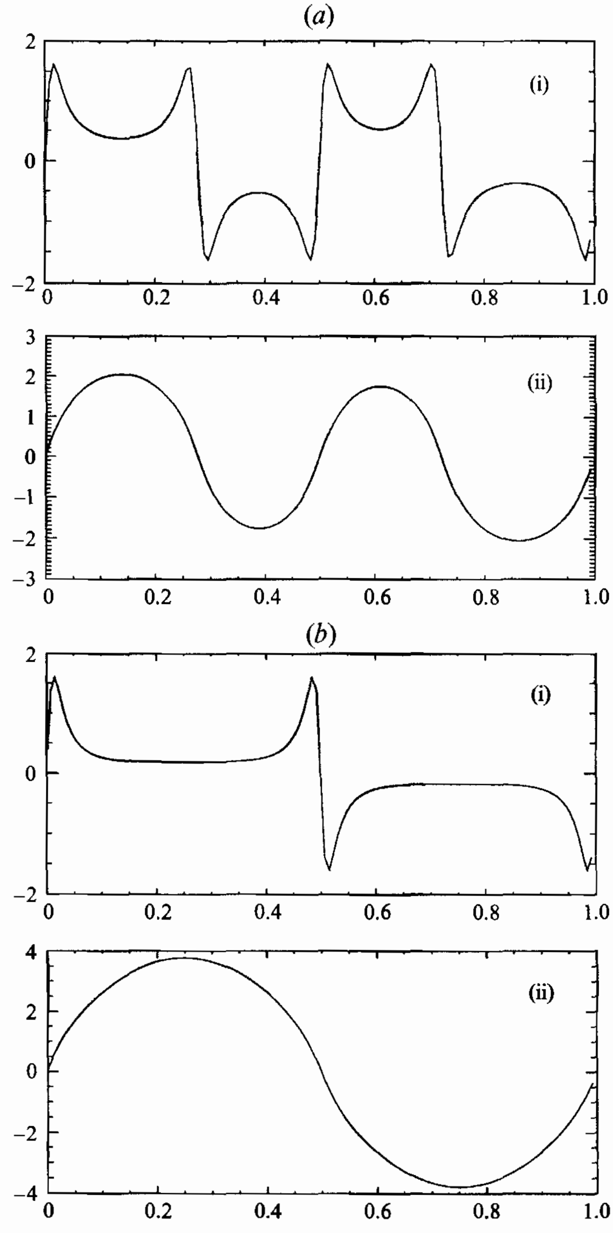

FIGURE 2. (*a*) Intermediate state of roll solution for $r = 10$. Note that this is not a steady solution since the functions are not yet symmetrical about all their extrema. (*b*) Final steady state for $r = 10$ showing the same wavelength as in the $r = 3$ case. The box length is again 2π. (i) c, (ii) w.

any case we rely on the result that vertically modulated modes are more stable on linearized theory than those studied here. The reduced system has made it possible to make predictions about the preferred planform of the instability, though at larger amplitudes only a fully two-dimensional numerical solution would yield definitive information. The system (3.14)–(3.15) can also be used to show that nonlinear wavelength selection acts to produce long horizontal scales, even if these are longer than the mode of maximum growth rate according to linearized theory.

Finally, we note that the methods used here could be extended to the other vertical temperature variations in the basic state, such as the central-layer type considered in BN, producing similar equations, though the calculations would be less straightforward and the coefficients different.

Helpful conversations with G. K. Batchelor are gratefully acknowledged. This work is supported by the UK SERC.

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